Keener, Theoretical Statistics
Chapter 4: Unbiased Estimation

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Four Key Properties of Normal Variables

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4. **Sums of Independent Normals**: If $X_1 \sim N(\mu_1, \sigma_1^2)$ and $X_2 \sim N(\mu_2, \sigma_2^2)$ are independent, then $X_1 + X_2 \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$.

Proof Sketch: MGF of a Standard Normal

Property 2: $M_Z(u) = e^{u^2/2}$

$$
\begin{aligned}
M_Z(u) = E[e^{uZ}] &= \int_{-\infty}^{\infty} e^{uz} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(uz - \frac{z^2}{2}\right) dz \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}(z^2 - 2uz)\right) dz \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}(z^2 - 2uz + u^2 - u^2)\right) dz \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}(z-u)^2 + \frac{u^2}{2}\right) dz \\
&= e^{u^2/2} \int_{-\infty}^{\infty} \underbrace{\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(z-u)^2}{2}\right)}_{\text{This is the PDF of a } N(u,1) \text{ distribution}} dz \\
&= e^{u^2/2} \times 1 = e^{u^2/2} \quad \square
\end{aligned}
$$

The One-Sample Problem Setup

The Model

Let $X_1, X_2, \ldots, X_n$ be an i.i.d. random sample from $N(\mu, \sigma^2)$.

Our goal is to understand the distributions of the primary estimators for μ and σ^2:

- **Sample Mean**: $\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$
- **Sample Variance**: $S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2$

A Key Fact from Sufficiency

The pair $(\bar{X}, S^2)$ is a **complete sufficient statistic** for the parameter $\theta = (\mu, \sigma^2)$.

Complete Sufficiency of $(\bar{X}, S^2)$

For $X_1, \dots, X_n \sim N(\mu, \sigma^2)$, the joint density belongs to the exponential family with complete sufficient statistic

$$T = (T_1, T_2) = \left(\sum_{i=1}^{n} X_i, \; \sum_{i=1}^{n} X_i^2 \right).$$

- $T_1 = \sum_{i=1}^{n} X_i$ (sum of observations)
- $T_2 = \sum_{i=1}^{n} X_i^2$ (sum of squared observations)

Connection to $(\bar{X}, S^2)$

$$\bar{X} = \frac{T_1}{n}, \quad S^2 = \frac{T_2 - T_1^2/n}{n-1}.$$

$$T_1 = n\bar{X}, \quad T_2 = (n-1)S^2 + n\bar{X}^2.$$

Key Point

Since (T_1, T_2) is complete sufficient, and the transformation is one-to-one, the pair $(\bar{X}, S^2)$ is also a **complete sufficient statistic**.

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2. **Find the distribution of the average.** Now, let $Y = \sum X_i$. We know $\bar{X} = \frac{1}{n} Y$. We use the general linear transformation property: if $Y \sim N(\mu_Y, \sigma_Y^2)$, then $aY \sim N(a\mu_Y, a^2\sigma_Y^2)$.

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$$\bar{X} \sim N\left(\frac{1}{n}(n\mu), \left(\frac{1}{n}\right)^2 (n\sigma^2)\right)$$

Result: Distribution of $\bar{X}$

$$\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

The Gamma Distribution

To understand S^2, we first need to understand the Gamma distribution.

Definition

A random variable X has a Gamma distribution with **shape** $\alpha > 0$ and **scale** $\beta > 0$, denoted $X \sim \Gamma(\alpha, \beta)$, if its PDF is:

$$f(x; \alpha, \beta) = \frac{x^{\alpha-1} e^{-x/\beta}}{\beta^\alpha \Gamma(\alpha)}, \quad \text{for } x > 0$$

where $\Gamma(\alpha)$ is the Gamma function: $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt$.

Key Properties of the Gamma Function

- $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$
- $\Gamma(n) = (n-1)!$ for integer $n \geq 1$
- $\Gamma(1/2) = \sqrt{\pi}$

Properties of the Gamma Distribution

Moment Generating Function (MGF)

If $X \sim \Gamma(\alpha, \beta)$, its MGF is $M_X(u) = (1 - u\beta)^{-\alpha}$ for $u < 1/\beta$.

Additive Property

If $X_1 \sim \Gamma(\alpha_1, \beta)$ and $X_2 \sim \Gamma(\alpha_2, \beta)$ are independent, then:

$$X_1 + X_2 \sim \Gamma(\alpha_1 + \alpha_2, \beta)$$

This follows directly from MGFs: $M_{X_1+X_2}(u) = M_{X_1}(u)M_{X_2}(u)$.

The Chi-Square (χ^2) Distribution

A Special Case of the Gamma Distribution

Definition

The chi-square distribution with p degrees of freedom, χ^2_p, is the distribution of the sum of the squares of p independent standard normal random variables.

$$W = Z_1^2 + Z_2^2 + \cdots + Z_p^2 \sim \chi^2_p, \quad \text{where } Z_i \overset{\text{i.i.d.}}{\sim} N(0,1)$$

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Connecting χ^2 to Gamma

Let's find the MGF of a χ^2_1 variable, i.e., the MGF of Z^2 where $Z \sim N(0,1)$.

$$\begin{aligned} M_{Z^2}(u) = E[e^{uZ^2}] &= \int_{-\infty}^{\infty} e^{uz^2} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2(\frac{1}{2}-u)} dz = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2(1-2u)}{2}} dz \end{aligned}$$

Connecting χ^2 to Gamma

This integrand looks like a normal PDF, $N(0, \sigma_*^2)$ with $\sigma_*^2 = (1 - 2u)^{-1}$. The integral of a normal PDF (without the constant) is $\sqrt{2\pi\sigma_*^2}$.

$$M_{Z^2}(u) = \frac{1}{\sqrt{2\pi}}\sqrt{2\pi(1 - 2u)^{-1}} = (1 - 2u)^{-1/2}$$

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This is the MGF of a $\Gamma(\alpha = 1/2, \beta = 2)$ distribution! So $\chi_1^2 \equiv \Gamma(1/2, 2)$.

Using the additive property of Gamma variables:

$$\chi_p^2 = \sum_{i=1}^{p} Z_i^2 \sim \Gamma(p \cdot 1/2, 2) = \Gamma(p/2, 2)$$

The MGF of a χ_p^2 is therefore $M(u) = (1-2u)^{-p/2}$.

Distribution of Sample Variance

Let's define a standardized quantity related to S^2.

$$V := \frac{(n-1)S^2}{\sigma^2} = \frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i - \bar{X})^2 = \sum_{i=1}^{n}\left(\frac{X_i - \bar{X}}{\sigma}\right)^2$$

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Our goal is to show that $V \sim \chi^2_{n-1}$.

Let $Z_i = \frac{X_i - \mu}{\sigma}$ and $\bar{Z} = \frac{\bar{X}-\mu}{\sigma}$. Then $Z_i \sim N(0,1)$ and $\bar{Z} = \frac{1}{n}\sum Z_i$.

$$\begin{aligned}
V &= \sum_{i=1}^{n}\left(\frac{X_i - \mu - (\bar{X} - \mu)}{\sigma}\right)^2 = \sum_{i=1}^{n}(Z_i - \bar{Z})^2 \\
&= \sum_{i=1}^{n}(Z_i^2 - 2Z_i\bar{Z} + \bar{Z}^2) \\
&= \sum Z_i^2 - 2\bar{Z}\sum Z_i + \sum \bar{Z}^2 \\
&= \sum Z_i^2 - 2\bar{Z}(n\bar{Z}) + n\bar{Z}^2 \\
&= \sum_{i=1}^{n} Z_i^2 - n\bar{Z}^2
\end{aligned}$$

Inspecting Each Term

From the previous slide, we see that:

$$V = \sum_{i=1}^{n} Z_i^2 - n\bar{Z}^2 \implies \sum_{i=1}^{n} Z_i^2 = V + n\bar{Z}^2$$

Let's analyze each piece of this equation:

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$$(\sqrt{n}\bar{Z})^2 = n\bar{Z}^2 \sim \chi_1^2$$

Putting It All Together with MGFs

Our key identity is:

$$\underbrace{\sum_{i=1}^{n} Z_i^2}_{\sim\chi_n^2} = \underbrace{V}_{\text{dist. unknown}} + \underbrace{n\bar{Z}^2}_{\sim\chi_1^2}$$

Basu's Theorem

A crucial (and non-trivial) result states that for a random sample from a normal distribution, the sample mean $\bar{X}$ and the sample variance S^2 are **independent**.

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Since V is a function of only S^2, and $n\bar{Z}^2$ is a function of only $\bar{X}$, it follows that V and $n\bar{Z}^2$ are also independent.

Putting It All Together with MGFs

$$\underbrace{\sum_{i=1}^{n} Z_i^2}_{\sim\chi^2_n} = \underbrace{V}_{\text{dist. unknown}} + \underbrace{n\bar{Z}^2}_{\sim\chi^2_1}$$

For independent random variables A and B, $M_{A+B}(u) = M_A(u)M_B(u)$. Applying this to our identity:

$$M_{\sum Z_i^2}(u) = M_V(u) \cdot M_{n\bar{Z}^2}(u)$$

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We know the MGFs for the χ^2 terms:

$$(1-2u)^{-n/2} = M_V(u) \cdot (1-2u)^{-1/2}$$

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Solving for $M_V(u)$:

$$M_V(u) = \frac{(1-2u)^{-n/2}}{(1-2u)^{-1/2}} = (1-2u)^{-(n-1)/2}$$

The Final Result and Summary

The moment generating function we found for V is:

$$M_V(u) = (1 - 2u)^{-(n-1)/2}$$

This is precisely the MGF of a chi-square distribution with $n - 1$ degrees of freedom.

Result: Distribution of S^2

By the uniqueness of MGFs, we have proven that:

$$V = \frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$$

This along with $\bar{X} \sim N(\mu, \sigma^2/n)$ implicitly determines the joint distribution of $\bar{X}$ and S^2, because these two variables are independent.

Section 4.4 Normal One-Sample Problem – Estimation

Our Goal

To find Uniformly Minimum-Variance Unbiased Estimators (UMVUEs) by creating unbiased functions of the complete sufficient statistic $(\bar{X}, S^2)$.

1. The General Formula for Estimating Powers of σ
2. Applications: Finding UMVUEs for σ^2 and σ
3. Finding UMVUEs for μ and μ^2
4. Other Examples: Signal-to-Noise and Quantiles

Expected Value of S^r

Using the fact that $V = \frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$, note that:

$$E[S^r] = \frac{\sigma^r 2^{r/2} \Gamma\left(\frac{r+n-1}{2}\right)}{(n-1)^{r/2} \Gamma\left(\frac{n-1}{2}\right)}$$

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This shows that S^r is a **biased** estimator for σ^r.

Constructing the UMVUE for σ^r

To make this unbiased, we multiply S^r by the reciprocal of the bias term. This gives the general formula for the UMVUE of σ^r:

$$\widehat{\sigma^r} = \frac{(n-1)^{r/2} \Gamma\left(\frac{n-1}{2}\right)}{2^{r/2} \Gamma\left(\frac{r+n-1}{2}\right)} S^r$$

This is guaranteed to be the UMVUE because it's an unbiased function of the complete sufficient statistic $(\bar{X}, S^2)$.

Application 1: UMVUE for the Variance σ^2

We can find the UMVUE for σ^2 by setting $r = 2$ in our general formula.

Setting $r = 2$

$$
\begin{aligned}
\widehat{\sigma^2} &= \frac{(n-1)^{2/2}\Gamma\left(\frac{n-1}{2}\right)}{2^{2/2}\Gamma\left(\frac{2+n-1}{2}\right)} S^2 \\
&= \frac{(n-1)\Gamma\left(\frac{n-1}{2}\right)}{2\Gamma\left(\frac{n+1}{2}\right)} S^2
\end{aligned}
$$

Using the property of the Gamma function, $\Gamma(x+1) = x\Gamma(x)$, we have:

$$
\Gamma\left(\frac{n+1}{2}\right) = \Gamma\left(\frac{n-1}{2} + 1\right) = \left(\frac{n-1}{2}\right)\Gamma\left(\frac{n-1}{2}\right)
$$

Application 1: UMVUE for the Variance σ^2

Simplifying the Estimator

$$\widehat{\sigma^2} = \frac{(n-1)\Gamma\left(\frac{n-1}{2}\right)}{2 \cdot \left(\frac{n-1}{2}\right)\Gamma\left(\frac{n-1}{2}\right)} S^2$$
$$= \frac{n-1}{n-1} S^2 = S^2$$

Conclusion

For $r = 2$, the correction factor simplifies to 1. S^2 is the UMVUE for σ^2.

Application 2: UMVUE for the Standard Deviation σ

To find the UMVUE for σ, we set $r = 1$ in the general formula.

Setting $r = 1$

$$\widehat{\sigma^1} = \frac{(n-1)^{1/2}\Gamma\left(\frac{n-1}{2}\right)}{2^{1/2}\Gamma\left(\frac{1+n-1}{2}\right)} S^1$$

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The UMVUE for σ

Simplifying the expression gives the UMVUE for σ:

$$\hat{\sigma} = \frac{\sqrt{n-1}\Gamma\left(\frac{n-1}{2}\right)}{\sqrt{2}\Gamma\left(\frac{n}{2}\right)} S$$

Note that the correction factor is not equal to 1, which confirms that S itself is a biased estimator for σ.

Estimating Functions of the Mean

UMVUE for the Mean, μ

Because $E[\bar{X}] = \mu$ and $\bar{X}$ is a function of the complete sufficient statistic, $\bar{X}$ is the UMVUE for μ.

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Estimating the Squared Mean, μ^2

The estimator $\bar{X}^2$ is a natural choice, but it is biased:

$$E[\bar{X}^2] = \mathrm{Var}(\bar{X}) + (E[\bar{X}])^2 = \frac{\sigma^2}{n} + \mu^2$$

The bias is σ^2/n.

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The bias can be removed by subtracting an unbiased estimate of σ^2/n. We know S^2 is the UMVUE for σ^2, so we use S^2/n.

$$\widehat{\mu^2} = \bar{X}^2 - \frac{S^2}{n}$$

This estimator is a function of $(\bar{X}, S^2)$ and is unbiased, making it the UMVUE for μ^2.

Other UMVUE Examples

Signal-to-Noise Ratio: μ/σ

The parameter μ/σ can be written as $\mu \cdot \sigma^{-1}$. Because $\bar{X}$ and S^2 are independent, the UMVUE for the product is the product of the individual UMVUEs.

$$\widehat{\mu/\sigma} = (\text{UMVUE for } \mu) \times (\text{UMVUE for } \sigma^{-1})$$

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Using the general formula with $r = -1$ for σ^{-1} gives:

$$\widehat{\mu/\sigma} = \bar{X} \cdot \left(\frac{\sqrt{2}\Gamma\left(\frac{n-1}{2}\right)}{S\sqrt{n-1}\Gamma\left(\frac{n-2}{2}\right)} \right)$$

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The p^{th} Quantile: $x_p = \mu + \sigma\Phi^{-1}(p)$

The quantile is a linear combination of μ and σ. The UMVUE is found by plugging in the UMVUEs for each component:

$$\widehat{x_p} = (\text{UMVUE for } \mu) + (\text{UMVUE for } \sigma) \cdot \Phi^{-1}(p)$$